# 1 Introduction

Previously, most endgame theory has been informal go theory. Apart from combinatorial game theory applied to the microendgame and excursions of thermography, we have known little more than the count and move value of a local endgame. Bill Spight's pioneer work into the structure of relating local values to each other and studying global model environments has enabled me to unfold this new approach into the powerful, consistent and revolutionary endgame theory in this book. Besides, it is very well applicable for us go players.

In the chapters *3 Local Endgame* and *4 Long Sequences,* we learn to evaluate every local endgame correctly. In the chapters *5 Playing in an Environment* to *10 Several Local Endgames Each with One Follow-up,* we study the global move order during the early or late endgame to decide in which local endgame to play next. The chapter *2 Basics* provides some necessary low level theory. The chapter *11 Scoring* relates area and territory scores as well as definitions of life.

We analyse a local endgame to determine its count, move value and gains in the initial position. While a simple local endgame has sequences of one or two plays, it is more difficult to evaluate a local endgame with long sequences. Apart from kos and exceptional types, 'local gote' and 'local sente' are the basic types of a local endgame. They have gote counts and move values versus sente counts and move values, respectively. We must determine the right values to avoid mistakes when choosing moves. Informal go theory has just guessed the type of a local endgame so wrong values could be determined and mistakes could occur.

Instead, our theory also determines the correct type. Depending on the kind of a simple local endgame, we can choose our favourite condition from up to four equivalent conditions. Besides, we establish equivalence of Black's and White's value perspectives, and the relations between counts, move values, gains or net profits. Traditional go theory must change as we prove non-existence of a local double sente.

Evaluation of a local endgame with long sequences requires iteration and benefits from simplification. The means of simplification include: dominating options, reversal and playing the difference game; comparison of two particular counts or move values; traversal of a sequence due to a comparison of its gains to a move value. After simplification, we evaluate a local endgame with long sequences like a simple local endgame.

Informal go theory often made the wrong assumption that the global move order could always be decided by playing in order of decreasing local move values. Instead, our theory for the global move order decides whether the correct next play is in a considered local endgame or in the global environment. For this purpose, we consider an alternating sum, such as $\Delta T$, of all move values in the environment during the late endgame to make an exact decision. During the early endgame, our decision is a good approximation typically depending on the *temperature* T, which is the largest move value in the environment, and our estimate of the value T/2 of first playing in it. We decide by comparing such global values to local values, such as the move value M and follow-up move value F of the local endgame.

For the global move order during the early or late endgame, we consider an environment when studying the basic kinds of local endgames with one player's follow-up, both players' follow-ups, or a player's alternative gote or sente options. We also touch the decision-making among several local endgames.

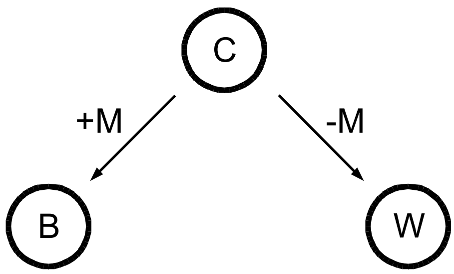
The global move order depends on being in the early versus late endgame, the starting player making his follow-up available and the temperature being 'high' (larger) versus 'low' (smaller than the follow-up move values). For many decisions, we avoid too complex tactical reading by providing these three equivalent fast methods: applying a principle, comparing two or three counts, or comparing two net profits. We enable further simplifications by ignoring superfluous values or cases.

In conclusion, we do not just fill a few gaps of informal go theory; instead, we invent a well developed new endgame theory of evaluation and decision-making on the local and global scales. We establish the theory as mathematical theorems and their proofs.

Unlike combinatorial game theory for the microendgame, the theory in this book allows large move values. Unlike the theoretically more powerful theory of thermography with its general definitions of 'mean values' and 'local temperatures', our consideration of 'counts' and 'move values' is more applicable during our games.

We ignore *equal options,* which can sometimes allow alternative strategies but are irrelevant for the basic structure of the proofs. The book does not include research on hyperactive kos and dame ko fights because this deserves an extra book. Future research should study: a) the close relation between traversal, reversal and difference games, their missing identity, and existence of pathological examples for which they differ; b) more theory on kos and threats than in the literature; c) detailed theory on difference games; d) more relations to combinatorial game theory and infinitesimals.

Bill Spight started mathematical research long ago and I continued it in winter 2016/7 and 2021. While I could sometimes prove up to four theorems of intermediate difficulty per day, every hard proof required a few days, and each of theorems 55 and 58 three weeks of work. See also *13.2 Inventors* (p. 237).




*simple gote without follow-up*

A *simple gote without follow-up* is a local endgame with the settled black child with the count $B$ and settled white child with the count $W$. An unsettled initial position implies $B > W$.

***Definitions 11*** *[gote values]*

A simple gote without follow-up has the

*gote count* $C := C_{GOTE} := (B + W) / 2$,

*gote move value* $M := M_{GOTE} := (B - W) / 2$.

*Remarks*

A gote count is calculated as the average of the followers' counts. A gote move value is calculated as half their difference value. If the initial position is unsettled, we have $B > C > W$. Except for a settled local endgame, a simple gote without follow-up is the simplest kind of a 'local gote'.

***Definitions 12*** *[gains]*

*Black's gain* is $G_B := B - C$, where $B$ is the count of the black child,

*White's gain* is $G_W := C - W$, where $W$ is the count of the white child.

*Remarks*

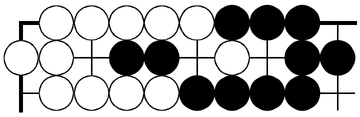
These definitions apply to all positions and follow-ups of all local endgames. The inconsistent *traditional endgame theory* does not use the division by 2 and therefore discourages all advanced endgame theory. The *modern endgame theory* in this book uses the division by 2 in the definition of 'gote move value' so that the theory is consistent. In particular, for a simple gote, the move value, Black's gain and White's gain are equal:

***Proposition 10*** *[equality of move value and gains in a simple gote]*

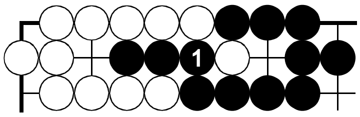
In a simple gote, $M = G_B = G_W$.

*Proof:* $G_B = B - C = B - (B + W) / 2 = (B - W) / 2 = M = (B - W) / 2 = (B + W) / 2 - W = C - W = G_W$.□

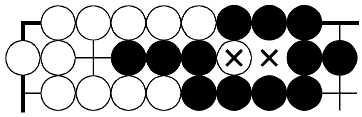
*Remark:* Proposition 10 is also a corollary of theorem 19.



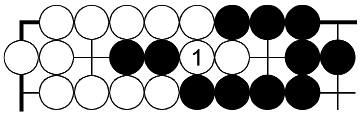
*Example 1: C = -1, M = 4*



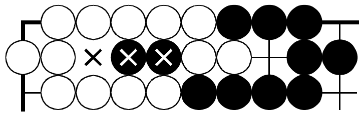
*Dia. 1.1:* $G_B = 4$



*Dia. 1.2: B = 3*



*Dia. 1.3:* $G_W = 4$



*Dia. 1.4: W = -5*

*Example 1:* In the initial unsettled position, the local endgame is a simple gote without follow-up. The black child, which is the direct follower after Black's one move, in *Dia. 1.2* is a settled position with the local count B = 3, which is Black's 3 local points minus White's 0 local points. The white child, which is the direct follower after White's one move, in *Dia. 1.4* is a settled position with the count W = -5, which is Black's 0 local points minus White's 5 local points. The initial local endgame has the gote count C = (B + W) / 2 = (3 + (-5)) / 2 = -2/2 = -1 and gote move value M = (B - W) / 2 = (3 - (-5)) / 2 = 8/2 = 4. Black 1 in *Dia. 1.1* gains $G_B = B - C = 3 - (-1) = 4$. White 1 in *Dia. 1.3* gains $G_W = C - W = -1 - (-5) = 4$. We confirm $M = G_B = G_W = 4$ (see proposition 10) and B > C > W <=> 3 > -1 > -5. For example, the initial local endgame has the white-count $\overline{C} = -C = -(-1) = 1$.

*Presuppositions*

Suppose the starting player's value perspective, simple gotes without follow-ups and with the move values $T_1 \geq T_2 \geq ... \geq T_N \geq 0$.

***Theorem 11** [decreasing order]*

Playing in order of decreasing-or-constant move values is correct.

*Proof*

Trivial for N = 1.

Now, let us prove for all natural numbers K, N with $1 < K \leq N$.

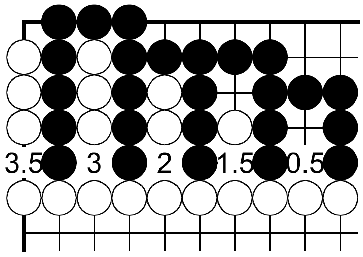
We study the only differing moves 1 and K as follows. Let $A_2, A_3,..., A_N$ be arbitrary indices so that $\{T_{A2},..., T_{AK},..., T_{AN}\} = \{T_2, T_3,..., T_N\}$, alternating sequences $S(\mathbf{T_1}, T_{A2}, T_{A3},..., T_{AK-1}, \mathbf{T_{AK}}, T_{AK+1},..., T_{AN})$ and $S'(\mathbf{T_{AK}}, T_{A2}, T_{A3},..., T_{AK-1}, \mathbf{T_1}, T_{AK+1},..., T_{AN})$.

*Case K odd:*

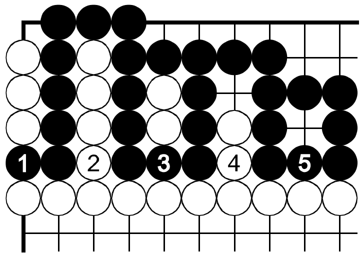
The starting player gains both $T_1$ and $T_{AK}$ so S and S' result in equal scores. Without loss of generality, choose S.

*Case K even:*

In S, the starting player gains $T_1 - T_{A_K} \geq 0$. In S', the starting player gains $T_{A_K} - T_1 \leq 0$. Hence, choosing S is at least as good as choosing S'.□




*Example 2: move values*




*Dia. 2.1: correct*

*Example 2:* It is correct to play the simple gotes without follow-ups in order of their decreasing-or-constant move values 3.5, 3, 2, 1.5, 0.5.

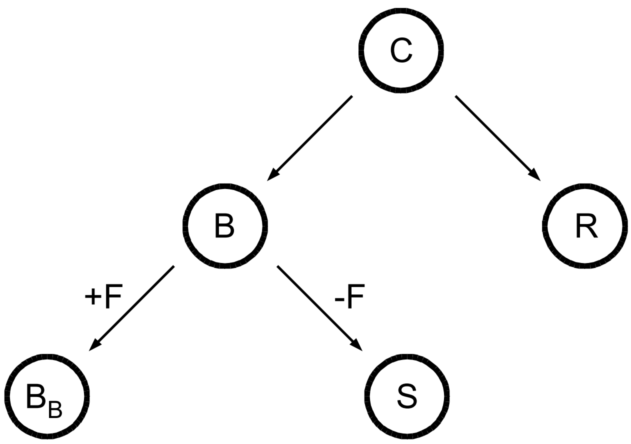
## 3.2 Black's Follow-up

*Introduction*

We study the theory of *Volume 3* chapter 3.3 about a local endgame with Black's simple follow-up. Its conditions compare tentative values, which we define. Conditions 1 compare sente count to gote count, $C_{SENTE}$ ? $C_{GOTE}$, conditions 2 compare gote move value to follow-up move value, $M_{GOTE}$ ? F, conditions 3 compare sente move value to follow-up move value, $M_{SENTE}$ ? F, conditions 4 compare sente move value to gote move value, $M_{SENTE}$ ? $M_{GOTE}$. Propositions 12 - 14 relate the conditions so theorem 15 can state their equivalence, we can define the types of the aforementioned kind of local endgames and determine their correct values.

*Basic Definitions*

A player's *gote sequence* is an alternating sequence of an odd number of moves started and ended by him. A player's *sente sequence* is an alternating sequence of an even number of moves started by him and ended by the opponent.




*local endgame with Black's follow-up*

*Remarks*

The propositions in the sections *3.2 Black's Follow-up* to *3.7 White's Options* are for single moves, but future research and application can be extended to gote sequences between the positions. Then, we would presume a) traversal of the sequences from the initial position to B or S and b) no necessity for more detailed calculations due to rapidly changing move values or complications due to intermediate plays elsewhere on the board. Unsettled initial position and black follower imply $B_B > S > R$.

*Presuppositions and Basic Definitions*

We have Black's local endgame with the initial position with the count C and move value M to be determined, the count R of the white child and 'gote or reverse sente' follower, the count $B_B$ of the follow-up's black child, the count $B_W = S$ of the follow-up's white child and 'sente sequence's' follower, the gote count $B := (B_B + B_W) / 2$, and gote move value $F := (B_B - B_W) / 2$ of the black child and follow-up.

***Definitions 13** [tentative values]*

The initial position of the local endgame has the

tentative gote count $C_{GOTE} := (B + R) / 2$,

tentative sente count $C_{SENTE} := S$,

tentative gote move value $M_{GOTE} := (B - R) / 2$,

tentative sente move value $M_{SENTE} := S - R$.

*Remarks*

In this section, the sente move value is Black's sente move value.

First, these are tentative values. Later, when a local endgame's type is determined, its fitting tentative values become its values.

A comparison ? being =, >, < identifies the 'ambiguous', 'local gote', 'local sente' conditions, respectively. $\overline{?}$ inverts > and <.

Modern endgame theory evolves from the definitions of the tentative values.

***Proposition 12** [conditions 1 and 2]*

$C_{SENTE}$ ? $C_{GOTE}$ <=> $M_{GOTE}$ ? F.

*Proof*

$C_{SENTE}$ ? $C_{GOTE}$ <=>$^{(def)}$ S ? (B + R) / 2 <=> (B + R) / 2 $\overline{?}$ S <=> (B + R) / 2 $\overline{?}$ $B_W$ <=> B + R $\overline{?}$ $2B_W$ <=>$^{(*1)}$ -B - R ? $-2B_W$ <=> B - R ? 2B -

$2B_W <=> B - R ? (2B - B_W) - B_W <=>^{(*2)} B - R ? B_B - B_W <=> (B - R) / 2 ? (B_B - B_W) / 2 <=>^{(def)} M_{GOTE} ? F$.□

*1) Multiplication by $-1$ inverts the inequality sign.

*2) We apply $B = (B_B + B_W) / 2 <=> 2B = B_B + B_W <=> 2B - B_W = B_B$.

***Proposition 13*** *[conditions 1 and 3]*

$C_{SENTE} ? C_{GOTE} <=> M_{SENTE} ? F$.

*Proof*

$C_{SENTE} ? C_{GOTE} <=>^{(def)} S ? (B + R) / 2 <=> 2S ? B + R <=> S + S ? B + R <=> S - R ? B - S <=>^{(*)(def)} M_{SENTE} ? F$.□

*) This transformation is possible because $B - S = F <=> B - F = S$, where the move value $F$ at the position $B$ is subtracted because a) White gains so Black loses it and b) the count $B$ at the position $B$ is transformed into the count $S$ of $B$'s white follower $S$.

***Proposition 14*** *[conditions 1 and 4]*

$C_{SENTE} ? C_{GOTE} <=> M_{SENTE} ? M_{GOTE}$.

*Proof*

$C_{SENTE} ? C_{GOTE} <=>^{(def)} S ? (B + R) / 2 <=> 2S ? B + R <=> 2S - 2R ? B - R <=> S - R ? (B - R) / 2 <=>^{(def)} M_{SENTE} ? M_{GOTE}$.□

***Definitions 14*** *[basic types]*

For a local endgame with Black's follow-up, we define these types:

ambiguous $:<=> M_{SENTE} = M_{GOTE}$,

local gote $:<=> M_{SENTE} > M_{GOTE}$,

local sente $:<=> M_{SENTE} < M_{GOTE}$.

*Table 1*

| Type | Count | Move Value |
|---|---|---|
| ambiguous | $C := C_{SENTE} := C_{SENTE} = C_{GOTE} := C_{GOTE}$ | $M := M_{SENTE} := M_{SENTE} = M_{GOTE} := M_{GOTE}$ |
| local gote | $C := C_{GOTE} := C_{GOTE}$ | $M := M_{GOTE} := M_{GOTE}$ |
| local sente | $C := C_{SENTE} := C_{SENTE}$ | $M := M_{SENTE} := M_{SENTE}$ |

Calculus) for general explanations of thermography, its theorems, its proofs and graphical representations of thermographs.

By applying definitions 9 of count and move value in a presumed rich environment, thermography determines their correct values for all temperatures.

We also presume no kos now or later. Extensions of thermography to kos exist in the literature (for example, *Combinatorial Game Theory,* section VII.5 Komaster Thermography).

Since the thermograph is defined as the pair $(B_T(P), W_T(P))$, thermography is about computing $B_T(P)$ and $W_T(P)$ as functions of $T$ iteratively. Analogue to tactical reading, we start from the settled positions and proceed backwards. We also determine counts and move values of the position $P$ and its followers.

The mappings of the *walls* $B_T(P)$ and $W_T(P)$ are derived from the auxiliary mappings, called *scaffolds,* $\beta_T(P)$ and $\omega_T(P)$, respectively. $\beta_T(P)$ and $\omega_T(P)$ might overstretch and express either locally playing player's loss when $\beta_T(P) < \omega_T(P)$ for some ranges of $T$, for which we correct this by letting both mappings coincide. As a result, we have $B_T(P) \geq W_T(P)$ for all $T$.

In graphical representations, the x-axis represents decreasing counts (Black favours larger counts further leftwards while White favours smaller counts further rightwards) and the y-axis represents increasing temperatures $T$ from $0$ to $\infty$. See the literature for an extension to $-1$ to $\infty$.

Without kos and in the visual appearance of the aforementioned orientation of the coordinate system, Black's segments increase by $45°$ (for $a - T$ terms representing gote) or are vertical (for constant terms representing his sente at positive temperature). White's segments decrease by $45°$ (for $a + T$ terms representing gote) or are vertical (for constant terms representing his sente at positive temperature). Coinciding mappings have the vertical *mast* at $C_P$ for $T \geq M_P$, above which the players should play in the environment.

***Algorithm 44*** *[calculating a thermograph]*

Presume an unsettled local position without kos in a rich environment.

For each settled follower $x$, set $B_T(x) = W_T(x) = C_x = x$ and $M_x = 0$.

Iterate the following steps:

1. For each unsettled position $P$, calculate $\beta_T(P) = \max W_T(P_B) - T$.
2. For each unsettled position $P$, calculate $\omega_T(P) = \min B_T(P_W) + T$.
3. Determine $\min T =: M_P$ so that $\beta_T(P) = \omega_T(P) =: C_P$.
4. Set $B_T(P) = C_P$ if $T \geq M_P$ or $= \beta_T(P)$ if $T < M_P$.
5. Set $W_T(P) = C_P$ if $T \geq M_P$ or $= \omega_T(P)$ if $T < M_P$.

*Details*

Each unsettled position $P$ includes each unsettled follower and eventually the unsettled initial local position. For the current $P$, we first calculate Black's scaffold $\beta_T(P)$ and White's scaffold $\omega_T(P)$, second determine the move value $M_P$ and count $C_P$, third derive Black's wall $B_T(P)$ and White's wall $W_T(P)$.

Black's scaffold $\beta_T(P)$ is derived from $\max W_T(P_B)$ of all $P_B$. We account $-T$ for Black's move. Black maximises among all his options (available next moves) $P_B$. For Black's follower, White is the next moving player so we iterate by calling White's wall $W_T()$.

White's scaffold $\omega_T(P)$ is derived from $\min B_T(P_W)$ of all $P_W$. We account $+T$ for White's move. White minimises among all his options (available next moves) $P_W$. For White's follower, Black is the next moving player so we iterate by calling Black's wall $B_T()$.

Since $P$ is unsettled, the currently moving player has at least one available option. In the example below, there is always exactly one option so we forgo the $\max$ or $\min$ notation and step of Black maximising or White minimising, respectively.

The accounting of $-T$ for Black's move or $+T$ for White's move is consistent with the equations $B_T(P) = \max W_T(P_B) - T$ and $W_T(P) = \min B_T(P_W) + T$ in the remarks on definitions 23 enhanced by considering the maximum or minimum for several options, respectively. We can interpret this as a tax for moving locally in a rich environment of temperature $T$.

How do we actually calculate Black's scaffold $\beta_T(P)$ when deriving it from $\max W_T(P_B)$ of all $P_B$? If Black's follower $P_B$ is settled with the count $x$, we simply subtract $T$ so have $\beta_T(P) = x - T$. If Black's maximising follower $P_B$ is unsettled, we consider the previously determined $W_T(P_B)$ and modify it by subtracting $T$ for each of its temperature ranges and simplifying the resulting terms.

How do we actually calculate White's scaffold $\omega_T(P)$ when deriving it from $\min B_T(P_W)$ of all $P_W$? If White's follower $P_W$ is settled with the count $x$, we simply add $T$ so have $\omega_T(P) = x + T$. If White's minimising follower $P_W$ is unsettled, we consider the previously determined $B_T(P_W)$ and modify it by adding $T$ for each of its temperature ranges and simplifying the resulting terms.

The players alternate and we have linear equations. Therefore, each term is a number, a number plus $T$, a number minus $T$, or $+T - T$ (or vice versa) cancel each other so we have a number again. Terms do not become increasingly complicated. It is, however, possible that several temperature ranges occur. An iteration step might add another temperature range or decrease the number of cases for different temperature ranges.

We apply definitions 9 of count and move value to Black's and White's scaffolds: we determine the minimum temperature $T$, which is the move value $M_P$, so that they are equal: $\beta_T(P) = \omega_T(P)$, which is the count $C_P$.

How do we actually determine these values? We consider the temperature ranges of $\beta_T(P)$ and $\omega_T(P)$. We check their temperature ranges in increasing order, starting from the lowest common temperature range. We stop on finding the smallest temperature $T$ for which equality $\beta_T(P) = \omega_T(P)$ occurs. For each checked temperature range, we apply its explicit terms of $\beta_T(P)$ and $\omega_T(P)$ in the equation $\beta_T(P) = \omega_T(P)$, possibly transform to simplify and dissolve $T$, and verify whether the equation is fulfilled for at least one temperature of the currently checked temperature range. If it is fulfilled, the dissolved $T$ is the minimum temperature, for which equality $\beta_T(P) = \omega_T(P)$ occurs. Otherwise, we proceed likewise with the next temperature range.

For large enough temperatures $T \geq M_P$, we set Black's wall $B_T(P)$ and White's wall $W_T(P)$ equal to the count: $B_T(P) = W_T(P) = C_P$. For smaller temperatures $T < M_P$ of mandatory local play, Black's wall $B_T(P)$ is Black's scaffold $\beta_T(P)$, that is $B_T(P) = \beta_T(P)$, and White's wall $W_T(P)$ is White's scaffold $\omega_T(P)$, that is $W_T(P) = \omega_T(P)$.

If Black's scaffold $\beta_T(P)$ has the same term for all temperatures, Black's wall $B_T(P)$ inherits this term only for the lower temperature range $T < M_P$. If Black's scaffold $\beta_T(P)$ comprises different terms for different temperature ranges, Black's wall $B_T(P)$ inherits them for the lower temperature range. However, the new second-highest temperature range receives $M_P$ as its excluded upper bound.

If White's scaffold $\omega_T(P)$ has the same term for all temperatures, White's wall $W_T(P)$ inherits this term only for the lower temperature range $T < M_P$. If White's scaffold $\omega_T(P)$ comprises different terms for different temperature ranges, White's wall $W_T(P)$ inherits them for the lower temperature range. However, the new second-highest temperature range receives $M_P$ as its excluded upper bound.

In step 3 of the algorithm, the smallest $T$, which characterises equality, always exists. Due to continuity of either scaffold or wall, we may choose where to put equalities in the inequalities of the temperature ranges.

Francisco Criado has discovered the important *Example 1,* whose calculation I work out. We study it again in the following section. In the graphs, the left solid trajectory is Black's wall, the right solid trajectory is White's wall, the thick dash line is the upper part of Black's scaffold, the thin dash line is the upper part of White's scaffold, the upper solid line with arrow is the mast.

*Example 1*

$P = \{Q|-3\}$, $Q = \{8|R\}$, $R = \{S|-1\}$, $S = \{7|U\}$, $U = \{1|-3\}$.

*Iteration Step 1 for U:*

*Scaffolds:*

$\beta_T(U) = \beta_T(\{1|-3\}) = 1 - T.$

$\omega_T(U) = \omega_T(\{1|-3\}) = -3 + T.$

*Equality:*

$\beta_T(U) = \omega_T(U) \Leftrightarrow 1 - T = -3 + T \Leftrightarrow 4 = 2T \Leftrightarrow 2 = T$

is the minimum, for which $\beta_T(U)$ and $\omega_T(U)$ are equal, so the move value is $M_U = T = 2$ and the count is $C_U = \beta_2(U) = 1 - 2 = \omega_2(U) = -3 + 2 = -1.$

*Walls:*

$B_T(U) = B_T(\{1|-3\})$

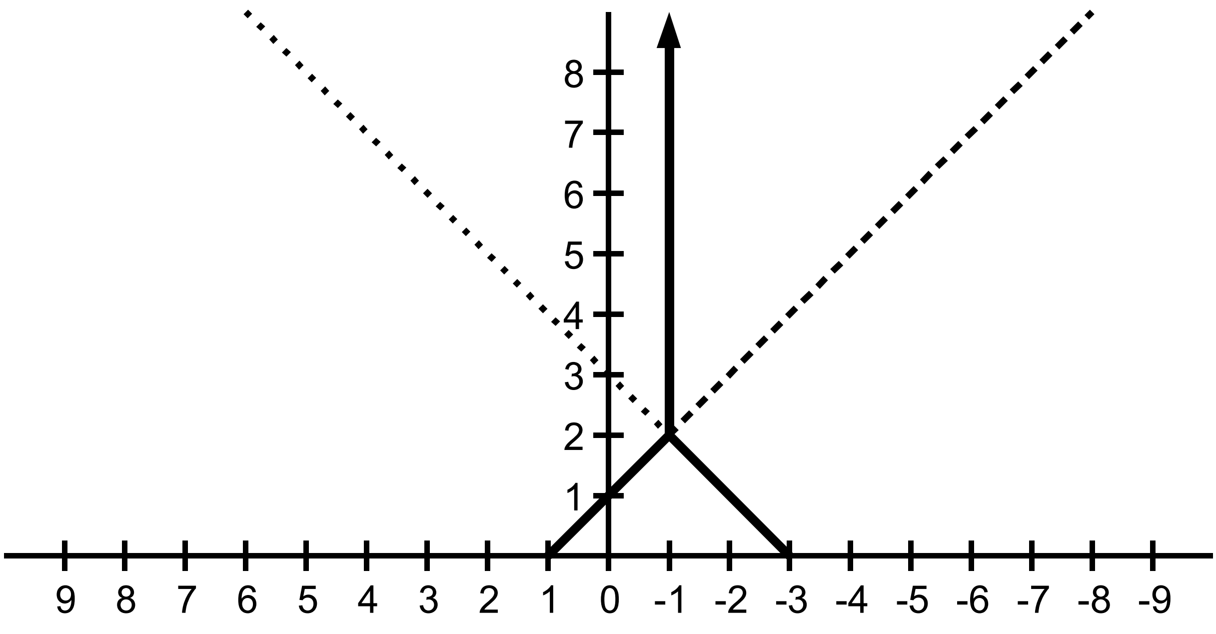
$= C_U = -1$ if $T \geq M_U = 2,$

$= \beta_T(U) = 1 - T$ if $T < 2.$

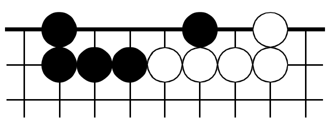
$W_T(U) = W_T(\{1|-3\})$

$= C_U = -1$ if $T \geq M_U = 2,$

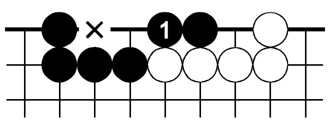
$= \omega_T(U) = -3 + T$ if $T < 2.$



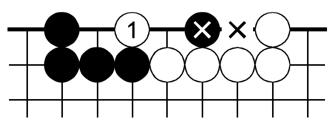

*Thermograph of U*



*Dia. 1.1: position U, $M_U = 2$, $C_U = -1$*



*Dia. 1.2: Black's score 1*



*Dia. 1.3: White's score -3*

*Iteration Step 2 for S:*

*Scaffolds:*

$\beta_T(S) = \beta_T(\{7|U\}) = 7 - T.$

$\omega_T(S) = \omega_T(\{7|U\}) = B_T(U) + T = B_T(\{1|-3\}) + T$

$= -1 + T$ if $T \geq 2$,

$= 1 - T + T = 1$ if $T < 2$.

*Equality:* For which minimum $T$ is $\beta_T(S) = \omega_T(S)$ fulfilled?

It is not fulfilled for $T < 2$ as $\beta_T(S) = \omega_T(S) \iff 7 - T = 1 \iff 6 = T$.

$\beta_T(S) = \omega_T(S) \iff 7 - T = -1 + T \iff 8 = 2T \iff 4 = T$

is the minimum, for which $\beta_T(S)$ and $\omega_T(S)$ with its $T \geq 2$ term are equal, so the move value is $M_S = T = 4$ and the count is $C_S = \beta_4(S) = 7 - 4 = \omega_4(S) = -1 + 4 = 3$.

*Walls:*

$B_T(S) = B_T(\{7|U\})$

$= C_S = 3$ if $T \geq M_S = 4$,

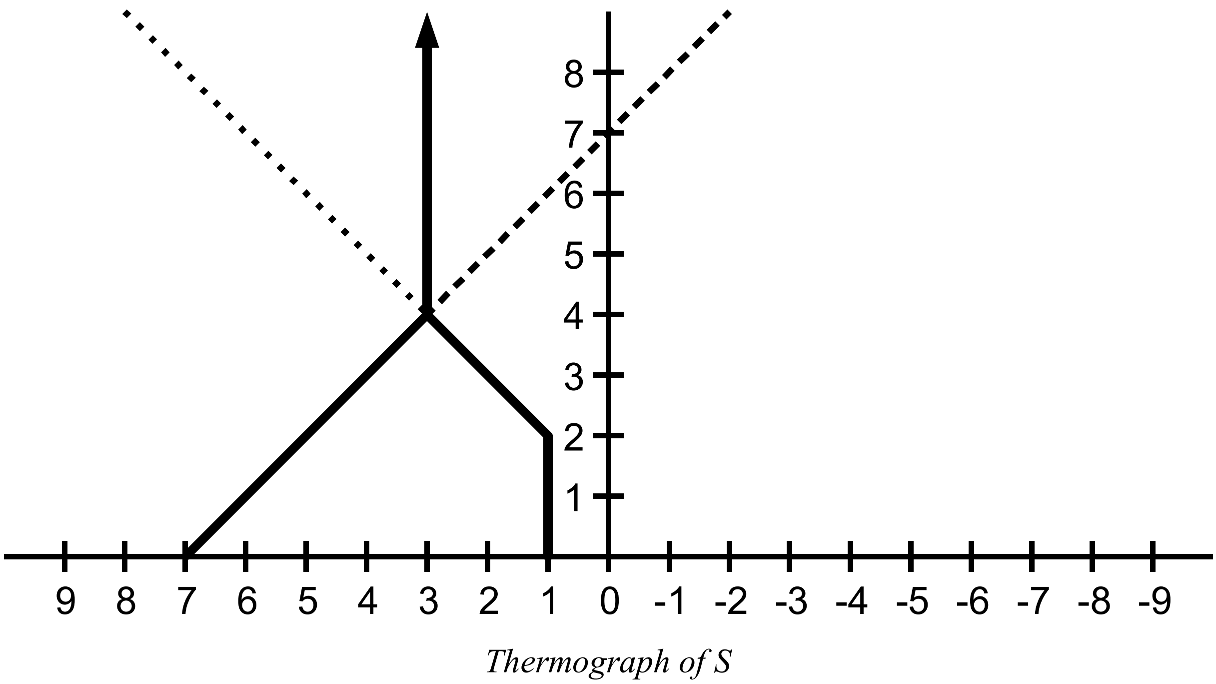
$= \beta_T(S) = 7 - T$ if $T < 4$.

$W_T(S) = W_T(\{7|U\})$

$= C_S = 3$ if $T \geq M_S = 4$,

$= \omega_T(\{7|U\}) = -1 + T$ if $4 > T \geq 2$,

$= \omega_T(\{7|U\}) = 1$ if $T < 2$.




*Thermograph of S*

*Example:* We have the ensemble of the two large local endgames A and B. The marked environment has the temperature T = 2 in its shown most valuable region. In *Dia. 1* with Black's correct start, the local count is 18 and Black gains about T/2 = 1 from starting in the environment so the result is $R_1 = 18 + 1 = 19$. In *Dia. 2,* the local count is 19, Black loses about T/2 = 1 as White starts in the environment so the result is $R_2 = 19 - 1 = 18$.

***Proposition 51*** *[maximum net profit of starting in a simple environment]*

T is the maximum net profit of starting in an environment of simple gotes without follow-ups and largest move value T.

*Proof*

Suppose the alternating sequence's counts $C_0, C_1, \ldots, C_N = 0$. By theorem 29 and its proof, the net profit is $-C_0$ as the sequence transforms $C_0 + (-C_0) = 0$.* If the sequence has only one move with the gain and move value $T = C_1 - C_0$, the transformation of counts is $C_0 + (-C_0) = C_1 = 0$. The aforementioned net profit is the maximum because, for several moves, the sequence's transformation is also (*).□

***Definitions 25*** *[alternating sum]*

ΔT is the finite sign-alternating sum of the values $T, T_1, \ldots$ in decreasing-or-constant order. ΔT|F is the sign-alternating sum of the values $T, T_1, \ldots$ and F considered together in their decreasing-or-constant order.

*Remarks*

An alternating sum of no values is 0. We can have $\Delta T = T - T_1 + \ldots$, $-\Delta T = -T + T_1 - \ldots$, $\Delta T_1 = T_1 - T_2 + \ldots$ or $\Delta T_V = T_V - T_{V-1} + \ldots$ One local follow-up move value, say, F can also become available in ΔT|F. For example, if the environment consists of the values 8, 6, 4, 2 and there is the local follow-up move value 7, then $\Delta T|F = 8 - 7 + 6 - 4 + 2$.

*Presuppositions*

Suppose A is set of numbers, $M \notin A$, $\overline{A} := \{a \in A \mid a > M\}$, $\underline{A} := \{a \in A \mid a < M\}$, $U := |\overline{A}|$.

***Proposition 52*** *[adding a unique value to an alternating sum]*

$\Delta A = \Delta\overline{A} + (-1)^U \Delta\underline{A}$ and

$\Delta A|M = \Delta\overline{A} + (-1)^U(M - \Delta\underline{A}) = \Delta\overline{A} + (-1)^U(\Delta\underline{A}|M)$.

## 8.3 Medium Temperature with Larger Follow-up

*Introduction*

The 'medium' temperature $T$ is smaller than the follow-up move value $E$ of the starting player and larger than the follow-up move value $F$ of the opponent. Proposition 92 started with two alternating sums in the term $\Delta T|E + \Delta T_1|F$, which proposition 97 slightly simplifies to $\Delta T_1 + \Delta T_1|F$. We define the alternating sum $\Lambda_F$ of the test sequences' intermediate parts. Proposition 98 states its equality to the simplified term. Now, theorem 99 uses a condition with the one alternating sum $\Lambda_F$ to decide whether to start in the environment or locally. Proposition 100 is a preparation for theorem 109, considers the two test sequences and relates their net profits to their gains. Compare *Volume 4* chapter 12.4.2.

***Proposition 97*** *[starting player's perspective, preparation for F < T < E]*

If $F < T < E$, the starting player starts

- in the environment if $2M_{GOTE} - E \leq \Delta T_1 + \Delta T_1|F$,
- locally if $2M_{GOTE} - E \geq \Delta T_1 + \Delta T_1|F$.

*Proof*

Comparison in proposition 92 $\Leftrightarrow$ (1) ? (2) $\Leftrightarrow$ $2M_{GOTE} - T \ ?\ \Delta T|E + \Delta T_1|F \Leftrightarrow^{(*)} 2M_{GOTE} - T \ ?\ E - \Delta T + \Delta T_1|F \Leftrightarrow 2M_{GOTE} - T \ ?\ E - T + \Delta T_1 + \Delta T_1|F \Leftrightarrow 2M_{GOTE} - E \ ?\ \Delta T_1 + \Delta T_1|F$. Accordingly, the maximising starting player starts in the environment if $(1) \leq (2)$ or locally if $(1) \geq (2)$.□

*) By assumption $E > T$.

***Definitions 31*** *[for F < T < E]*

$\mathbf{\Lambda_F}$ is the alternating sum of twice those move values of the environment excluding $T$ that are larger than $F$, and the follow-up $F$.

For $\Lambda_F$, the *intermediate part* of a sequence excludes $T$ and consists of the other moves of the environment with move values larger than $F$, and the follow-up $F$ if available.

*Remarks*

$\Lambda_F$ (pronounce: lamda F) evaluates the net profits of the intermediate parts. $F$ has a plus or minus sign for an even or odd number, respectively, of move values of the environment excluding $T$ that are larger than $F$.

***Proposition 98*** *[relation to alternating sum for F < T < E]*

$\Delta T_1 + \Delta T_1|F = \Lambda_F$.

*Proof*

*Case L even:*

$\Delta T_1 + \Delta T_1|F = (T_1 - T_2 + ... - T_L + T_{L+1} - T_{L+2} + ...) + (T_1 - T_2 + ... - T_L + F - T_{L+1} + T_{L+2} - ...) = 2(T_1 - T_2 + ... - T_L) + F = \Lambda_F$.

*Case L odd:*

$\Delta T_1 + \Delta T_1|F = (T_1 - T_2 + ... + T_L - T_{L+1} + T_{L+2} - ...) + (T_1 - T_2 + ... + T_L - F + T_{L+1} - T_{L+2} + ...) = 2(T_1 - T_2 + ... + T_L) - F = \Lambda_F$.□

***Theorem 99*** *[starting player's perspective and start for F < T < E]*

If $F < T < E$, the starting player starts

- in the environment if $2M_{GOTE} - E \leq \Lambda_F$,
- locally if $2M_{GOTE} - E \geq \Lambda_F$.

*Proof:* Apply proposition 98 to proposition 97.□

***Proposition 100*** *[starting player's perspective, difference for F < T < E]*

If $F < T < E$, then $P_2 - P_1 = E - 2M_{GOTE} + \Lambda_F$.

*Proof*

We have stated the sequences in the proof of proposition 92 as

1) $P_1 := M_{GOTE} - \Delta T|E$

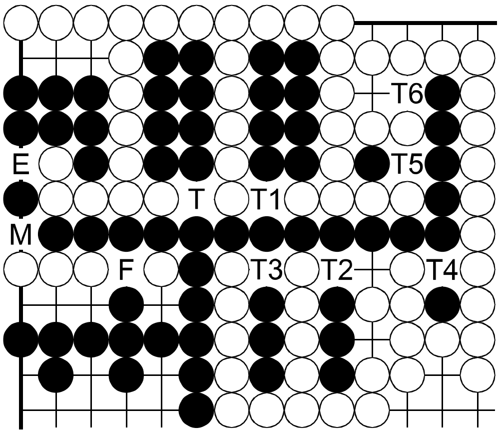
2) $P_2 := T - M_{GOTE} + \Delta T_1|F$

$P_2 - P_1 = (T - M_{GOTE} + \Delta T_1|F) - (M_{GOTE} - \Delta T|E) =^{(*1)} (T - M_{GOTE} + \Delta T_1|F) - (M_{GOTE} - E + \Delta T) = (T - M_{GOTE} + \Delta T_1|F) - (M_{GOTE} - E + T - \Delta T_1) = T - M_{GOTE} + \Delta T_1|F - M_{GOTE} + E - T + \Delta T_1 = E - 2M_{GOTE} + \Delta T_1 + \Delta T_1|F =^{(*2)} E - 2M_{GOTE} + \Lambda_F$.□

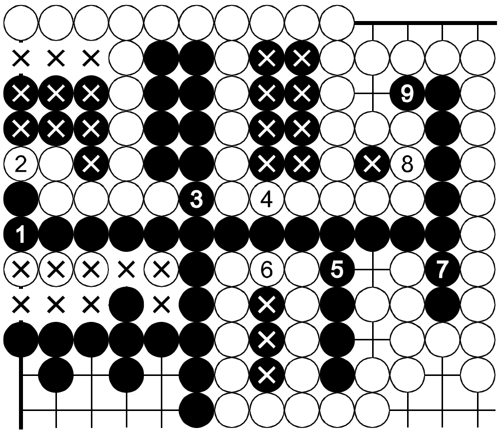
*1) By assumption $E > T$.

*2) By proposition 98.

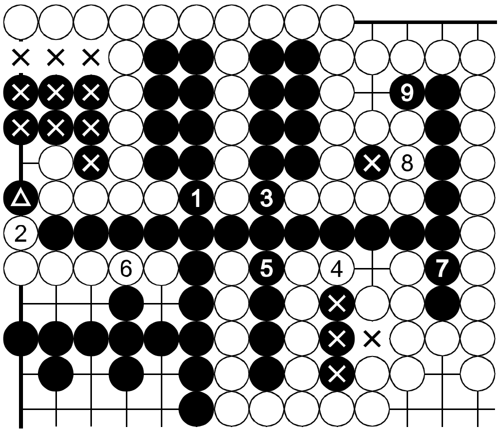
*Example 1:* We have: a local gote with the gote move value $M_{GOTE} = 10.5$, the follow-up move values $E = 8.5$ and $F = 1.5$; an environment with $T = 8$, $T_1 = 8$, $T_2 = 3.5$, $T_3 = 3$, $T_4 = 1$, $T_5 = 1$, $T_6 = 0.5$; $F < T < E \iff 1.5 < 8 < 8.5$, $\Lambda_F = 2*(T_1 - T_2 + T_3) - F = 2*(8 - 3.5 + 3) - 1.5 = 13.5$. According to theorem 99, Black starts in the environment at 'T' because $2M_{GOTE} - E < \Lambda_F \iff 2*10.5 - 8.5 < 13.5 \iff 12.5 < 13.5$. We have the counts $C_1 = -28$ and $C_2 = -27$, and net profits $P_1 = M_{GOTE} - E + T - T_1 + T_2 - T_3 + T_4 - T_5 + T_6 = 10.5 - 8.5 + 8 - 8 + 3.5 - 3 + 1 - 1 + 0.5 = 3$ and $P_2 = T - M_{GOTE} + T_1 - T_2 + T_3 - F + T_4 - T_5 + T_6 = 8 - 10.5 + 8 - 3.5 + 3 - 1.5 + 1 - 1 + 0.5 = 4$.




*Example 1: Black starts*



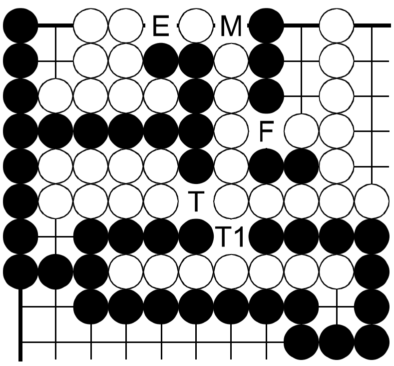
*Dia. 1.1: sequence 1, $C_1 = -28$, $P_1 = 3$*



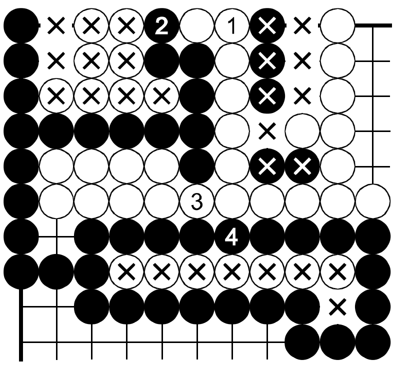
*Dia. 1.2: sequence 2, $C_2 = -27$, $P_2 = 4$*

Theorem 93 confirms Black's start in the environment because $P_1 \leq P_2 \iff 3 \leq 4$ and $C_1 \leq C_2 \iff -28 \leq -27$. According to definitions 34 and theorem 109, we have $C_2 - C_1 = P_2 - P_1 = \Lambda_F - (2M_{GOTE} - E) \iff -27 - (-28) = 4 - 3 = 13.5 - (2*10.5 - 8.5) = 1$.

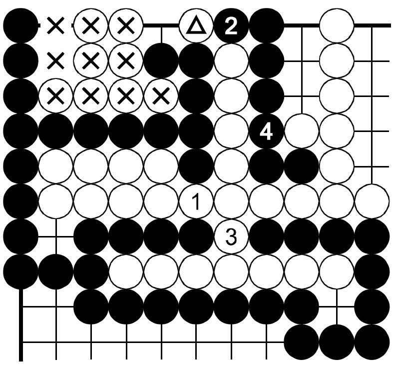
*Example 2:* We have a local gote with the gote move value $M_{GOTE} = 11$, the follow-up move values $E = 9$ and $F = 2$.




*Example 2: White starts*



*Dia. 2.1: sequence 1, $\overline{C}_1 = -19$, $\overline{P}_1 = 3$*



*Dia. 2.2: sequence 2, $\overline{C}_2 = -19$, $\overline{P}_2 = 3$*

We have: an environment with $T = 8.5$, $T_1 = 7.5$; $\Lambda_F = 2*T_1 - F = 2*7.5 - 2 = 13$, $F < T < E \iff 2 < 8.5 < 9$. According to theorem 99, White starts locally at 'M' or in the environment at 'T' because $2M_{GOTE} - E = \Lambda_F \iff 2*11 - 9 = 13 \iff 13 = 13$. We have the white-counts $\overline{C}_1 = \overline{C}_2 = -19$, and net white-profits $\overline{P}_1 = M_{GOTE} - E + T - T_1 = 11 - 9 + 8.5 - 7.5 = 3$ and $\overline{P}_2 = T - M_{GOTE} + T_1 - F = 8.5 - 11 + 7.5 - 2 = 3$. Theorem 93 confirms that White starts locally or in the environment since $\overline{P}_1 = \overline{P}_2 \iff 3 = 3$ and $\overline{C}_1 = \overline{C}_2 \iff -19 = -19$. According to definitions 34 and theorem 109, we have $\overline{C}_2 - \overline{C}_1 = \overline{P}_2 - \overline{P}_1 = \Lambda_F - (2M_{GOTE} - E) \iff -19 - (-19) = 3 - 3 = 13 - (2*11 - 9) = 0$.